Persistence of iterated partial sums

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March 4, 2013

Abstract

Let $S_n^{(2)}$ denote the iterated partial sums. That is, $S_n^{(2)} = S_1 + S_2 + \cdots + S_n$, where $S_i = X_1 + X_2 + \cdots + X_i$. Assuming $X_1, X_2, \ldots, X_n$ are integrable, zero-mean, i.i.d. random variables, we show that the persistence probabilities

$$p_n^{(2)} := \mathbb{P}\left(\max_{1 \le i \le n} S_i^{(2)} < 0\right) \le c\sqrt{\frac{\mathbb{E}|S_{n+1}|}{(n+1)\mathbb{E}|X_1|}},$$

with $c \le 6\sqrt{30}$ (and $c = 2$ whenever X_1 is symmetric). The converse inequality holds whenever the non-zero $\min(-X_1, 0)$ is bounded or when it has only finite third moment and in addition X_1 is squared integrable. Furthermore, $p_n^{(2)} \asymp n^{-1/4}$ for any non-degenerate squared integrable, i.i.d., zero-mean X_i. In contrast, we show that for any $0 < \gamma < 1/4$ there exist integrable, zero-mean random variables for which the rate of decay of $p_n^{(2)}$ is $n^{-\gamma}$.

1 Introduction

The estimation of probabilities of rare events is one of the central themes of research in the theory of probability. Of particular note are *persistence* probabilities, formulated as

$$q_n = \mathbb{P}\Big(\max_{1 \le k \le n} Y_k < y\Big), \tag{1}$$

where $\{Y_k\}_{k=1}^n$ is a sequence of zero-mean random variables. For independent Y_i the persistence probability is easily determined to be the product of $\mathbb{P}(Y_k < y)$ and to a large extent this extends to the case of sufficiently weakly dependent and similarly distributed Y_i, where typically q_n decays exponentially in n. In contrast, in the classical case of partial sums, namely $Y_k = S_k = \sum_{i=1}^k X_i$ with $\{X_j\}$ i.i.d. zero-mean random variables, it is well known that $q_n = O(n^{-1/2})$ decays as a power law. This seems to be one of the very few cases in which a power law decay for q_n can be proved and its exponent is explicitly known. Indeed, within the large class of similar problems where dependence between Y_i is strong enough to rule out exponential decay, the behavior of q_n is very sensitive to the precise structure of dependence between the variables Y_i and even merely determining its asymptotic rate can be very challenging (for example, see

*The research was supported in part by NSF grants DMS-0806211 and DMS-1106627.

†This research partially done when Jian Ding was at MSRI program in 2012.

‡This research partially done when Fuchang Gao visited Stanford university in March 2010. The authors also thank NSF grant DMS-0806211 for the support.

[4] for recent results in case $Y_k = \sum_{i=1}^{n} X_i(1-c_{k,n})^i$ are the values of a random Kac polynomials evaluated at certain non-random $\{c_{k,n}\}$).

We focus here on iterated sums of i.i.d. zero-mean, random variables $\{X_i\}$. That is, with $S_n = \sum_{k=1}^{n} X_k$ and

$$S_n^{(2)} = \sum_{k=1}^{n} S_k = \sum_{i=1}^{n} (n-i+1)X_i\,, \tag{2}$$

we are interested in the asymptotics as $n \to \infty$ of the persistence probabilities

$$p_n^{(2)}(y) := \mathbb{P}\left(\max_{1\le k\le n} S_k^{(2)} < y\right),\ \overline{p}_n^{(2)}(y) := \mathbb{P}\left(\max_{1\le k\le n} S_k^{(2)} \le y\right), \tag{3}$$

where $y \ge 0$ is independent of n. With $y \ll n$ it immediately follows from Lindeberg's CLT (when X_i are square integrable), that $p_n^{(2)}(y) \to 0$ as $n \to \infty$ and our goal is thus to find a sharp rate for this decay to zero.

Note that for any fixed $y > 0$ we have that $\overline{p}_n^{(2)}(y) \asymp p_n^{(2)}(y) \asymp p_n^{(2)}(0)$ up to a constant depending only on y, here and throughout the paper, $A \asymp B$ means that there exists two positive constants C_1 and C_2, such that $C_1 A \le B \le C_2 A$. Indeed, because $\mathbb{E}X_1^- > 0$, clearly $\mathbb{P}(X_1 < -\varepsilon) > 0$ for $\varepsilon = y/k$ and some integer $k \ge 1$. Now, for any $n \ge 1$ and $z \ge 0$,

$$\overline{p}_n^{(2)}(z) \ge p_n^{(2)}(z) \ge \mathbb{P}(X_1 < -\varepsilon)\overline{p}_{n-1}^{(2)}(z+\varepsilon) \ge \mathbb{P}(X_1 < -\varepsilon)\overline{p}_n^{(2)}(z+\varepsilon)$$

and applying this inequality for $z = i\varepsilon$, $i = 0, 1, \dots, k-1$ we conclude that

$$p_n^{(2)}(0) \ge [\mathbb{P}(X_1 < -\varepsilon)]^k \overline{p}_n^{(2)}(y). \tag{4}$$

Of course, we also have the complementary trivial relations $p_n^{(2)}(0) \le \overline{p}_n^{(2)}(0) \le p_n^{(2)}(y) \le \overline{p}_n^{(2)}(y)$, so it suffices to consider only $p_n^{(2)}(0)$ and $\overline{p}_n^{(2)}(0)$ which we denote hereafter by $p_n^{(2)}$ and $\overline{p}_n^{(2)}$, respectively. Obviously, $p_n^{(2)}$ and $\overline{p}_n^{(2)}$ have the same order (with $p_n^{(2)} = \overline{p}_n^{(2)}$ whenever X_1 has a density), and we consider both only in order to draw the reader's attention to potential identities connecting the two sequences $\{p_n^{(2)}\}$ and $\{\overline{p}_n^{(2)}\}$.

Persistence probabilities such as $p_n^{(2)}$ appear in many applications. For example, the precise problem we consider here arises in the study of the so-called sticky particle systems (c.f. [12] and the references therein). In case of standard normal X_i it is also related to entropic repulsion for ∇^2-Gaussian fields (c.f. [3] and the references therein), though here we consider the easiest version, namely a one dimensional ∇^2-Gaussian field. In his 1992 seminal paper, Sinai [11] proved that if $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$, then $p_n^{(2)} \asymp n^{-1/4}$. However, his method relies on the fact that for Bernoulli $\{X_k\}$ all local minima of $k \mapsto S_k^{(2)}$ correspond to values of k where $S_k = 0$, and as such form a sequence of regeneration times. For this reason, Sinai's method can not be readily extended to most other distributions. Using a different approach, more recently Vysotsky [13] managed to extend Sinai's result that $p_n^{(2)} \asymp n^{-1/4}$ to X_i which are double-sided exponential, and a few other special types of random walks. At about the same time, Aurzada and Dereich [2] used strong approximation techniques to prove the bounds $n^{-1/4}(\log n)^{-4} \lesssim p_n^{(2)} \lesssim n^{-1/4}(\log n)^4$ for zero-mean random variables $\{X_i\}$ such that $\mathbb{E}[e^{\beta|X_1|}] < \infty$ for some $\beta > 0$. However, even for X_i which are standard normal variables it was not known before the present results whether these logarithmic terms are needed, and if not, how to get rid of them. Our main result, stated below, fully resolves this question, requiring only that $\mathbb{E}X_1^2$ is finite and positive.

Theorem 1.1. *For i.i.d. $\{X_k\}$ of zero mean and $0 < \mathbb{E}|X_1| < \infty$, let $S_n^{(2)} = S_1 + S_2 + \cdots + S_n$, where $S_i = X_1 + X_2 + \cdots + X_i$. Then,*

$$\sum_{k=0}^{n} p_k^{(2)} \overline{p}_{n-k}^{(2)} \le c_1^2 \frac{\mathbb{E}|S_{n+1}|}{\mathbb{E}|X_1|}, \tag{5}$$

where $c_1 \le 6\sqrt{30}$, and $c_1 = 2$ if X_1 is symmetric. The converse inequality

$$\sum_{k=0}^{n} p_k^{(2)} p_{n-k}^{(2)} \ge \frac{1}{c_2} \frac{\mathbb{E}|S_{n+1}|}{\mathbb{E}|X_1|} \tag{6}$$

holds for some finite c_2 whenever X_1^- is bounded, or with X_1^- only having finite third moment and X_1 squared integrable. Taken together, these bounds imply that

$$\frac{1}{4c_1c_2} \sqrt{\frac{\mathbb{E}|S_{n+1}|}{(n+1)\mathbb{E}|X_1|}} \le p_n^{(2)} \le c_1 \sqrt{\frac{\mathbb{E}|S_{n+1}|}{(n+1)\mathbb{E}|X_1|}}. \tag{7}$$

Furthermore, assuming only *that $\mathbb{E}X_1 = 0$ and $0 < \mathbb{E}(X_1^2) < \infty$, we have that*

$$p_n^{(2)} \asymp n^{-1/4}. \tag{8}$$

Remark 1.2. *In contrast to (8), for any $0 < \gamma < 1/4$ there exists integrable, zero-mean variable X_1 for which $p_n^{(2)} \asymp n^{-\gamma}$. Indeed, considering $\mathbb{P}(Y_1 > y) = y^{-\alpha} 1_{y \ge 1}$ with $1 < \alpha < 2$, the bounds (7) hold for the bounded below, zero-mean, integrable random variable $X_1 = Y_1 - \mathbb{E}Y_1$. Setting $a_n = n^{1/\alpha}$, clearly $n\mathbb{P}(|X_1| > a_n x) \to x^{-\alpha}$ as $n \to \infty$, hence $a_n^{-1} S_n - b_n$ converges in distribution to a zero-mean, one-sided Stable$_\alpha$ variable Z_α, and it is further easy to check that $b_n = a_n^{-1} n \mathbb{E}[X_1 1_{|X_1| \le a_n}] \to b_\infty = -\mathbb{E}Y_1$. In fact, it is not hard to verify that $\{a_n^{-1} S_n\}$ is a uniformly integrable sequence and consequently $n^{-1/\alpha}\mathbb{E}|S_n| \to \mathbb{E}|Z_\alpha - \mathbb{E}Y_1|$ finite and positive. From Theorem 1.1 we then deduce that $p_n^{(2)} \asymp n^{-\gamma}$ for $\gamma = (1 - 1/\alpha)/2$. This rate matches with the corresponding one for integrated Lévy α-stable process, c.f. [10].*

The sequences $\{S_k\}$ and $\{S_k^{(2)}\}$ are special cases of the class of auto-regressive processes $Y_k = \sum_{\ell=1}^{L} a_\ell Y_{k-\ell} + X_k$ with zero initial conditions, i.e. $Y_k \equiv 0$ when $k \le 0$ (where S_k corresponds to $L = a_1 = 1$ and $S_k^{(2)}$ corresponds to $L = a_1 = 2$, $a_2 = -1$). While for such stochastic processes $(Y_k, \dots, Y_{k-L+1})$ is a time-homogeneous Markov chain of state space $\mathbb{R}^L$ and $q_n = \mathbb{P}(\tau > n)$ is merely the upper tail of the first hitting time τ of $[y, \infty)$ by the first coordinate of the chain, the general theory of Markov chains does not provide the precise decay of q_n, which even in case $L = 1$ ranges from exponential decay for $a_1 > 0$ small enough (which can be proved by comparing with O-U process, c.f. [1]), via the $O(n^{-1/2})$ decay for $a = 1$ to a constant $n \mapsto q_n$ in the limit $a_1 \uparrow \infty$. While we do not pursue this here, we believe that the approach we develop for proving Theorem 1.1 can potentially determine the asymptotic behavior of q_n for a large collection of auto-regressive processes. This is of much interest, since for example, as shown in [7], the asymptotic tail probability that random Kac polynomials have no (or few) real roots is determined in terms of the limit as $r \to \infty$ of the power law tail decay exponents for the iterates $S_k^{(r)} = \sum_{i=1}^{k} S_i^{(r-1)}$, $r \ge 3$.

Our approach further suggests that there might be some identities connecting the sequences $\{p_n^{(2)}\}$ and $\{\overline{p}_n^{(2)}\}$. Note that, if we denote

$$p_n^{(1)} = \mathbb{P}\left(\max_{1 \le k \le n} S_k < 0\right), \quad \overline{p}_n^{(1)} = \mathbb{P}\left(\max_{1 \le k \le n} S_k \le 0\right),$$

then as we show in the proof of the following proposition that there are indeed identities connecting the sequences $\{p_n^{(1)}\}$ and $\{\overline{p}_n^{(1)}\}$.

Proposition 1.3. *If X_i are mean zero i.i.d. symmetric random variables then for all $n \geq 1$,*

$$p_n^{(1)} \leq \frac{(2n-1)!!}{(2n)!!} \leq \overline{p}_n^{(1)}\,. \tag{9}$$

In particular, if X_1 also has a density, then

$$p_n^{(1)} = \frac{(2n-1)!!}{(2n)!!}\,. \tag{10}$$

Remark 1.4. *Proposition 1.3 is not new and can be found in [6, Section XII.8]. In fact, it is shown there that for all zero-mean random variables with bounded second moment (not necessary symmetric),*

$$p_n^{(1)} \asymp n^{-1/2}\,. \tag{11}$$

The novel point is our elegant proof, which serves as the starting point of our approach to the study of $p_n^{(2)}$.

Remark 1.5. *Let $B(s)$ denote a Brownian motion starting at $B(0) = 0$ and consider the integrated Brownian motion $Y(t) = \int_0^t B(s)ds$. Sinai [11] proved the existence of positive constants A_1 and A_2 such that for any $T \geq 1$,*

$$A_1 T^{-1/4} \leq \mathbb{P}\Big(\sup_{t\in[0,T]} Y(t) \leq 1\Big) \leq A_2 T^{-1/4}. \tag{12}$$

Upon setting $\varepsilon = T^{-3/2}$ and $t = uT$, by Brownian motion scaling this is equivalent up to a constant to the following result that can be derived from an implicit formula of McKean [9] (c.f. [5]):

$$\lim_{\varepsilon\to 0^+} \varepsilon^{-1/6}\mathbb{P}\Big(\sup_{u\in[0,1]} Y(u) \leq \varepsilon\Big) = \frac{3\Gamma(5/4)}{4\pi\sqrt{2\sqrt{2\pi}}}.$$

Since the iterated partial sums $S_n^{(2)}$ corresponding to i.i.d. standard normal random variables $\{X_i\}$, forms a "discretization" of $Y(t)$, the right-most inequality in (12) readily follows from Theorem 1.1. Indeed, with $\mathbb{E}[Y(k)Y(m)] = k^2(3m-k)/6$ and $\mathbb{E}[S_k^{(2)} S_m^{(2)}] = k(k+1)(3m-k+1)/6$ for $m \geq k$, setting $Z(k) = \sqrt{(1+1/k)(1+1/(2k)}Y(k)$, results with $\mathbb{E}[(S_k^{(2)})^2] = \mathbb{E}[Z(k)^2]$ and it is further not hard to show that $f(m,k) := \mathbb{E}[S_m^{(2)} S_k^{(2)}]/\mathbb{E}[Z(m)Z(k)] \geq 1$ for all $m \neq k$ (as $f(k+1,k) \geq 1$ and $df(m,k)/dm > 0$ for any $m \geq k+1$). Thus, by Slepian's lemma, we have that for any y

$$\mathbb{P}\Big(\max_{1\leq k\leq n} Z(k) < y\Big) \leq p_n^{(2)}(y)\,,$$

and setting n as the integer part of $T \geq 1$ it follows that

$$\mathbb{P}\Big(\sup_{t\in[0,T]} Y(t) \leq 1\Big) \leq \mathbb{P}\Big(\max_{1\leq k\leq n} Y(k) \leq 1\Big) \leq \mathbb{P}\Big(\max_{1\leq k\leq n} Z(k) < 2\Big) \leq p_n^{(2)}(2)\,.$$

Since $p_n^{(2)}(2) \leq c p_n^{(2)}$ for some finite constant c and all n, we conclude from Theorem 1.1 that

$$\mathbb{P}\Big(\sup_{t\in[0,T]} Y(t) \leq 1\Big) \leq 2c(n+1)^{-1/4} \leq 2cT^{-1/4}\,.$$

2 Proof of Proposition 1.3

Setting $S_0 = 0$ let $M_n = \max_{0 \le j \le n} S_j$ and consider the $\{0, 1, 2, \ldots, n\}$-valued random variable

$$\mathcal{N} = \min\{l \ge 0 : S_l = M_n\} \ .$$

For each $k = 1, 2, \ldots, n-1$ we have that

$$\begin{aligned}\{\mathcal{N} = k\} = \{&X_k > 0, X_k + X_{k-1} > 0, \ldots, X_k + X_{k-1} + \cdots + X_1 > 0;\\ &X_{k+1} \le 0, X_{k+1} + X_{k+2} \le 0, \ldots, X_{k+1} + X_{k+2} + \cdots + X_n \le 0\}.\end{aligned}$$

By the independence of $\{X_i\}$, the latter identity implies that

$$\begin{aligned}\mathbb{P}(\mathcal{N} = k) &= \mathbb{P}(X_k > 0, X_k + X_{k-1} > 0, \ldots, X_k + X_{k-1} + \cdots + X_1 > 0)\\ &\quad \times \mathbb{P}(X_{k+1} \le 0, X_{k+1} + X_{k+2} \le 0, \ldots, X_{k+1} + X_{k+2} + \cdots + X_n \le 0)\\ &= p_k^{(1)} \overline{p}_{n-k}^{(1)},\end{aligned}$$

where the last equality follows from our assumptions that X_i are i.i.d. symmetric random variables. Also note that $\mathbb{P}(\mathcal{N} = 0) = \overline{p}_n^{(1)}$ and

$$\mathbb{P}(\mathcal{N} = n) = \mathbb{P}(X_n > 0, X_n + X_{n-1} > 0, \ldots, X_n + X_{n-1} + \cdots + X_1 > 0) = p_n^{(1)}.$$

Thus, setting $p_0^{(1)} = \overline{p}_0^{(1)} = 1$ we arrive at the identity

$$\sum_{k=0}^{n} p_k^{(1)} \overline{p}_{n-k}^{(1)} = \sum_{k=0}^{n} \mathbb{P}(\mathcal{N} = k) = 1, \tag{13}$$

holding for all $n \ge 0$.

Fixing $x \in [0, 1)$, upon multiplying (13) by x^n and summing over $n \ge 0$, we arrive at $P(x)\overline{P}(x) = \frac{1}{1-x}$, where $P(x) = \sum_{k=0}^{\infty} p_k^{(1)} x^k$ and $\overline{P}(x) = \sum_{k=0}^{\infty} \overline{p}_k^{(1)} x^k$. Now, if X_1 also has a density then $p_k^{(1)} = \overline{p}_k^{(1)}$ for all k and so by the preceding $P(x) = \overline{P}(x) = (1-x)^{-1/2}$. Consequently, $p_n^{(1)}$ is merely the coefficient of x^n in the Taylor expansion at $x = 0$ of the function $(1-x)^{-1/2}$, from which we immediately deduce the identity (10).

If X_1 does not have a density, let $\{Y_i\}$ be i.i.d. standard normal random variables, independent of the sequence $\{X_i\}$ and denote by S_k and $\tilde{S}_k$ the partial sums of $\{X_i\}$ and $\{Y_i\}$, respectively. Note that for any $\varepsilon > 0$, each of the i.i.d. variables $X_i + \varepsilon Y_i$ is symmetric and has a density, with the corresponding partial sums being $S_k + \varepsilon \tilde{S}_k$. Hence, for any $\delta > 0$ we have that

$$\begin{aligned}\mathbb{P}\Big(\max_{1 \le k \le n} S_k < -\delta\Big) &\le \mathbb{P}\Big(\max_{1 \le k \le n} (S_k + \varepsilon \tilde{S}_k) \le 0\Big) + \mathbb{P}\Big(\max_{1 \le k \le n} \varepsilon \tilde{S}_k \ge \delta\Big)\\ &= \frac{(2n-1)!!}{(2n)!!} + \mathbb{P}\Big(\max_{1 \le k \le n} \varepsilon \tilde{S}_k \ge \delta\Big).\end{aligned}$$

Taking first $\varepsilon \downarrow 0$ followed by $\delta \downarrow 0$, we conclude that

$$\mathbb{P}\Big(\max_{1 \le k \le n} S_k < 0\Big) \le \frac{(2n-1)!!}{(2n)!!} \ ,$$

and a similar argument works for the remaining inequality in (9).

3 Proof of Theorem 1.1

By otherwise considering $X_i/\mathbb{E}|X_i|$, we assume without loss of generality that $\mathbb{E}|X_1| = 1$. To adapt the method of Section 2 for dealing with the iterated partial sums $S_n^{(2)}$, we introduce the parameter $t \in \mathbb{R}$ and consider the iterates $S_j^{(2)}(t) = S_0(t) + \cdots + S_j(t)$, $j \geq 0$, of the translated partial sums $S_k(t) = t + S_k$, $k \geq 0$. That is, $S_j^{(2)}(t) = (j+1)t + S_j^{(2)}$ for each $j \geq 0$.

Having fixed the value of t, we define the following $\{0, 1, 2, \dots, n\}$-valued random variable

$$\mathcal{K}_t = \min\Big\{ l \geq 0 : S_l^{(2)}(t) = \max_{0 \leq j \leq n} S_j^{(2)}(t) \Big\}.$$

Then, for each $k = 2, 3, \dots, n-2$, we have the identity

$$\begin{aligned}
\{\mathcal{K}_t = k\} = \{&S_k(t) > 0, S_k(t) + S_{k-1}(t) > 0, \dots, S_k(t) + S_{k-1}(t) + \cdots + S_1(t) > 0; \\
&S_{k+1}(t) \leq 0, S_{k+1}(t) + S_{k+2}(t) \leq 0, \dots, S_{k+1}(t) + S_{k+2}(t) + \cdots + S_n(t) \leq 0\} \\
= \{&S_k(t) > 0; X_k < 2S_k(t), \dots, (k-1)X_k + \cdots + X_2 < kS_k(t)\} \cap \{S_{k+1}(t) \leq 0\} \\
\cap \{&X_{k+2} \leq -2S_{k+1}(t), \dots, (n-k-1)X_{k+2} + \cdots + X_n \leq -(n-k)S_{k+1}(t)\}\,.
\end{aligned}$$

Next, for $2 \leq k \leq n$ we define $Y_{k,2} \in \sigma(X_2, \dots, X_k)$ and $Y_{k,n} \in \sigma(X_k, \dots, X_n)$ such that

$$\begin{aligned}
Y_{k,2} &= \max\left\{ \frac{X_k}{2}, \frac{2X_k + X_{k-1}}{3}, \dots, \frac{(k-1)X_k + \cdots + X_2}{k} \right\}, \\
Y_{k,n} &= \max\left\{ \frac{X_k}{2}, \frac{2X_k + X_{k+1}}{3}, \dots, \frac{(n-k+1)X_k + \cdots + X_n}{n-k+2} \right\}.
\end{aligned}$$

It is then not hard to verify that the preceding identities translate into

$$\begin{aligned}
\{\mathcal{K}_t = k\} &= \{S_k(t) > 0 \geq S_{k+1}(t)\} \cap \{Y_{k,2} < S_k(t)\} \cap \{Y_{k+2,n} \leq -S_{k+1}(t)\} &\quad (14)\\
&= \{-S_k + (Y_{k,2})^+ < t \leq -X_{k+1} - S_k - (Y_{k+2,n})^+\} &\quad (15)
\end{aligned}$$

holding for each $k = 2, \dots, n-2$. Further, for $k = 1$ and $k = n-1$ we have that

$$\begin{aligned}
\{\mathcal{K}_t = 1\} &= \{S_1(t) > 0\} \cap \{S_2(t) \leq 0\} \cap \{Y_{3,n} \leq -S_2(t)\}\,, \\
\{\mathcal{K}_t = n-1\} &= \{S_{n-1}(t) > 0\} \cap \{Y_{n-1,2} < S_{n-1}(t)\} \cap \{S_n(t) \leq 0\}\,,
\end{aligned}$$

so upon setting $Y_{1,2} = Y_{n+1,n} = -\infty$, the identities (14) and (15) extend to all $1 \leq k \leq n-1$.

For the remaining cases, that is, for $k = 0$ and $k = n$, we have instead that

$$\begin{aligned}
\{\mathcal{K}_t = 0\} &= \{t \leq -X_1 - (Y_{2,n})^+\}\,, &\quad (16)\\
\{\mathcal{K}_t = n\} &= \{-S_n + (Y_{n,2})^+ < t\}\,. &\quad (17)
\end{aligned}$$

In contrast with the proof of Proposition 1.3, here we have events $\{(Y_{k,2})^+ < S_k(t)\}$ and $\{(Y_{k+2,n})^+ \leq -S_{k+1}(t)\}$ that are linked through $S_k(t)$ and consequently not independent of each other. Our goal is to unhook this relation and in fact the parameter t was introduced precisely for this purpose.

3.1 Upper bound

For any integer $n > 1$, let

$$A_n = \max_{1\le k\le n}\{-S_{k+1}\}, \quad B_n = -\max_{1\le k\le n}\{S_k\}.$$

By definition $A_n \ge B_n$. Further, for any $1 \le k \le n-1$, from (14) we have that the event $\{\mathcal{K}_t = k\}$ implies that $\{S_k(t) > 0 \ge S_{k+1}(t)\} = \{-S_k < t \le -S_{k+1}\}$ and hence that $\{B_{n-1} < t \le A_{n-1}\}$. From (15) we also see that for any $1 \le k \le n-1$,

$$\int_{\mathbb{R}} 1_{\{\mathcal{K}_t=k\}} dt \ge (X_{k+1})^- 1_{\{Y_{k,2}<0\}} 1_{\{Y_{k+2,n}\le 0\}}$$

and consequently,

$$\begin{aligned} A_{n-1} - B_{n-1} = \int_{\mathbb{R}} 1_{\{B_{n-1}<t\le A_{n-1}\}} dt &\ge \sum_{k=1}^{n-1} \int_{\mathbb{R}} 1_{\{\mathcal{K}_t=k\}} dt \\ &\ge \sum_{k=1}^{n-1} (X_{k+1})^- 1_{\{Y_{k,2}<0\}} 1_{\{Y_{k+2,n}\le 0\}}\,. \end{aligned} \tag{18}$$

Taking the expectation of both sides we deduce from the mutual independence of $Y_{k,2}$, X_{k+1} and $Y_{k+2,n}$ that

$$\mathbb{E}[A_{n-1} - B_{n-1}] \ge \sum_{k=1}^{n-1} \mathbb{E}[(X_{k+1})^-]\mathbb{P}(Y_{k,2} < 0)\mathbb{P}(Y_{k+2,n} \le 0)\,.$$

Next, observe that since the sequence $\{X_i\}$ has an exchangeable law,

$$\begin{aligned} \mathbb{P}(Y_{k,2} < 0) &= \mathbb{P}(X_k < 0, 2X_k + X_{k-1} < 0, \dots, (k-1)X_k + \cdots + X_2 < 0) \\ &= \mathbb{P}(X_1 < 0, 2X_1 + X_2 < 0, \dots, (k-1)X_1 + \cdots + X_{k-1} < 0) = p^{(2)}_{k-1}\,. \end{aligned} \tag{19}$$

Similarly, $\mathbb{P}(Y_{k+2,n} \le 0) = \overline{p}^{(2)}_{n-1-k}$. With X_{k+1} having zero mean, we have that $\mathbb{E}[(X_{k+1})^-] = \mathbb{E}[(X_{k+1})^+] = 1/2$ (by our assumption that $\mathbb{E}|X_{k+1}| = \mathbb{E}|X_1| = 1$). Consequently, for any $n > 2$,

$$\mathbb{E}[A_{n-1} - B_{n-1}] \ge \frac{1}{2}\sum_{k=1}^{n-1} p^{(2)}_{k-1}\overline{p}^{(2)}_{n-1-k} = \frac{1}{2}\sum_{k=0}^{n-2} p^{(2)}_{k}\overline{p}^{(2)}_{n-2-k}\,.$$

With $\mathbb{E}[S_{n+1}] = 0$ and $\{X_k\}$ exchangeable, we clearly have that

$$\mathbb{E}[A_n - B_n] = \mathbb{E}[\max_{1\le k\le n}\{S_{n+1} - S_{k+1}\}] + \mathbb{E}[\max_{1\le k\le n} S_k] = 2\mathbb{E}[\max_{1\le k\le n} S_k]\,. \tag{20}$$

Recall Ottaviani's maximal inequality that for a symmetric random walk $\mathbb{P}(\max_{k=1}^n S_k \ge t) \le 2\mathbb{P}(S_n \ge t)$ for any $n, t \ge 0$, hence in this case

$$\mathbb{E}[\max_{1\le k\le n} S_k] \le 2\int_0^\infty \mathbb{P}(S_n \ge t)dt = \mathbb{E}|S_n|\,.$$

To deal with the general case, we replace Ottaviani's maximal inequality by Montgomery-Smith's inequality

$$\mathbb{P}(\max_{1\le k\le n}|S_k| \ge t) \le 3\max_{1\le k\le n}\mathbb{P}(|S_k| \ge t/3) \le 9\mathbb{P}(|S_n| \ge t/30)$$

(see [8]), from which we deduce that

$$\mathbb{E}[\max_{1\le k\le n} S_k] \le 9\int_0^\infty \mathbb{P}(|S_n|\ge t/30)dt = 270\mathbb{E}|S_n| \tag{21}$$

and thereby get (5). Finally, since $n\mapsto p_n^{(2)}$ is non-increasing and $p_n^{(2)} \le \overline{p}_n^{(2)}$, the upper bound of (7) is an immediate consequence of (5).

3.2 Lower bound

Turning to obtain the lower bound, let

$$m_n := -X_1 - (Y_{2,n})^+ \,, \quad M_n := -S_n + (Y_{n,2})^+ \,.$$

Note that for any $n\ge 2$, by using the last term of the maxima in the definition of $Y_{n,2}$ and $Y_{2,n}$, we have

$$Y_{n,2}+Y_{2,n} \ge \frac{1}{n}[(n-1)X_n+\cdots+X_2] + \frac{1}{n}[(n-1)X_2+\cdots+X_n] = S_n - X_1\,,$$

and consequently,

$$M_n - m_n \ge X_1 - S_n + (Y_{2,n}+Y_{n,2})^+ \ge (X_1-S_n)^+ = (X_2+\cdots+X_n)^-\,. \tag{22}$$

In particular, $M_n\ge m_n$. From (16) and (17) we know that if $m_n < t \le M_n$ then necessarily $1\le \mathcal{K}_t \le n-1$. Therefore,

$$M_n - m_n = \int_{\mathbb{R}} 1_{\{m_n<t\le M_n\}}dt \le \sum_{k=1}^{n-1}\int_{\mathbb{R}} 1_{\{\mathcal{K}_t=k\}}dt\,. \tag{23}$$

In view of (15) we have that for any $1\le k\le n-1$,

$$b_k := \mathbb{E}[\int_{\mathbb{R}} 1_{\{\mathcal{K}_t=k\}}dt] = \mathbb{E}\Big[\,\big(X_{k+1} + (Y_{k,2})^+ \,+ (Y_{k+2,n})^+\big)^-\,\Big].$$

By the mutual independence of the three variables on the right side, and since $\{X_k\}$ have identical distribution, we find that

$$\begin{aligned} b_k &= \int_0^\infty \mathbb{P}(X_{k+1}<-x, (Y_{k,2})^+ \,+ (Y_{k+2,n})^+ < x)dx \\ &\le \int_0^\infty \mathbb{P}(-X_1>x)\mathbb{P}(Y_{k,2}<x)\mathbb{P}(Y_{k+2,n}<x)dx\,. \end{aligned} \tag{24}$$

Next, setting $T^{(2)}_{i,k} = T_{1,k}+\cdots+T_{i,k}$ for $T_{i,k} = X_k+\cdots+X_{k+1-i}$, $i\ge 1$ and $T_{0,k}:=0$, observe that for any $0\le j\le k-1$ and $\ell\ge 1$,

$$T^{(2)}_{j+\ell,k+\ell} = T^{(2)}_{\ell-1,k+\ell} + (j+1)T_{\ell,k+\ell} + T^{(2)}_{j,k}\,.$$

Hence, with $A_{\ell,k} := \{T^{(2)}_{i,k}<0,\, i=1,\ldots,\ell-1\}$, just as we did in deriving the identity (19), we have that for any $\ell\ge 1$,

$$\begin{aligned} \{Y_{k+\ell,2}<0\} &= \{A_{\ell,k+\ell},\quad T^{(2)}_{\ell-1,k+\ell} + (j+1)T_{\ell,k+\ell} + T^{(2)}_{j,k} < 0,\, 0\le j\le k-1\}\,, \\ \{Y_{k,2}<x\} &= \{T^{(2)}_{j,k} < (j+1)x,\, 1\le j\le k-1\}\,. \end{aligned}$$

Consequently,
$$\{Y_{k,2} < x\} \bigcap \{T_{\ell,k+\ell} < -x\} \bigcap A_{\ell,k+\ell} \subseteq \{Y_{k+\ell,2} < 0\}\,.$$
By exchangeability of $\{X_m\}$ we have that for any k, ℓ,
$$\mathbb{P}(A_{\ell,k+\ell}) = p^{(2)}_{\ell-1} = \mathbb{P}(Y_{\ell,2} < 0)\,.$$
Thus, applying Harris's inequality for the non-increasing events $A_{\ell,k+\ell}$ and $\{T_{\ell,k+\ell} < -x\}$, we get by the independence of $\{X_m\}$ that
$$p^{(2)}_{k+\ell-1} = \mathbb{P}(Y_{k+\ell,2} < 0) \ge \mathbb{P}(Y_{k,2} < x)\mathbb{P}(T_{\ell,k+\ell} < -x) p^{(2)}_{\ell-1}\,.$$
Since $T_{\ell,k+\ell}$ has the same law as S_ℓ we thus get the bound
$$\mathbb{P}(Y_{k,2} < x) \le \frac{p^{(2)}_{k+\ell-1}}{p^{(2)}_{\ell-1}\mathbb{P}(S_\ell < -x)}\,,$$
for any $\ell \ge 1$. Similarly, we have that for any $\ell \ge 1$,
$$\mathbb{P}(Y_{k+2,n} < x) \le \frac{p^{(2)}_{n-k+\ell-1}}{p^{(2)}_{\ell-1}\mathbb{P}(S_\ell \le -x)}\,.$$
Clearly $k \mapsto p^{(2)}_k$ is non-increasing, so combining these bounds we find that
$$b_k \le \frac{c_2}{2} p^{(2)}_k p^{(2)}_{n-k}\,, \tag{25}$$
for $c_2 := 2\int_0^\infty \mathbb{P}(-X_1 > x) g(x)^{-2} dx$, where
$$g(x) := \sup_{\ell \ge 1} \{p^{(2)}_{\ell-1}\, \mathbb{P}(S_\ell < -x)\}\,. \tag{26}$$
For $(X_1)^-$ bounded it clearly suffices to show that $g(x) > 0$ for each fixed $x > 0$, and this trivially holds by the positivity of $\mathbb{P}(X_1 < -r)$ for $r > 0$ small enough (hence, $p^{(2)}_\ell \ge \mathbb{P}(X_1 < -r)^\ell$ also positive). Assuming instead that X_1 has finite (and positive) second moment, from (11) and the trivial bound $p^{(2)}_{\ell-1} \ge p^{(1)}_\ell$ we have that for some $\kappa > 0$ and all x,
$$g(x) \ge \kappa \sup_{\ell \ge 1}\{\frac{1}{\sqrt{\ell}}\, \mathbb{P}(S_\ell < -x)\}\,.$$
Further, by the CLT there exists $M < \infty$ large enough such that $\eta := \inf_{x>0} \mathbb{P}(S_{\lceil xM \rceil^2} < -x)$ is positive. Hence, setting $\ell = \lceil xM \rceil^2$, we deduce that in this case $g(x) \ge c/(1 + xM)$ for some $c > 0$ and all $x \ge 0$. Consequently, c_2 is then finite provided
$$3M \int_0^\infty (1 + xM)^2 \mathbb{P}(-X_1 > x) dx \le \mathbb{E}[(1 + (X_1)^- M)^3] < \infty\,,$$
i.e. whenever $(X_1)^-$ has finite third moment. Next, considering the expectation of both sides of (23) we deduce that under the above stated conditions, for any $n > 2$,
$$\mathbb{E}(M_n - m_n) \le \frac{c_2}{2} \sum_{k=1}^{n-1} p^{(2)}_{k-1} p^{(2)}_{n-k-1}\,.$$

In view of (22) we also have that $\mathbb{E}(M_n - m_n) \geq \mathbb{E}[(S_{n-1})^-] = \frac{1}{2}\mathbb{E}|S_{n-1}|$, from which we conclude that (6) holds for all $n \geq 1$.

Turning to lower bound $p_n^{(2)}$ as in (7), recall that $n \mapsto p_n^{(2)}$ is non-increasing. Hence, applying (6) for $n = 2m+1$ and utilizing the previously derived upper bound of (7) we have that

$$\begin{aligned}\frac{1}{c_2}\mathbb{E}|S_{2(m+1)}| &\leq 2\sum_{k=0}^{m} p_k^{(2)} p_m^{(2)} \leq 2c_1 p_m^{(2)} \sum_{k=0}^{m} \sqrt{\frac{\mathbb{E}|S_{k+1}|}{k+1}} \\ &\leq 4c_1 p_m^{(2)} \sqrt{(m+1)\mathbb{E}|S_{m+1}|}\,, \end{aligned} \tag{27}$$

where in the last inequality we use the fact that for independent, zero-mean $\{X_k\}$, the sequence $|S_k|$ is a sub-martingle, hence $k \mapsto \mathbb{E}|S_k|$ is non-decreasing. This proves the lower bound of (7).

Our starting point for removing in (8) the finite third moment assumption on $(X_1)^-$ is the following lemma which allows us to consider in the sequel only $k = O(n)$.

Lemma 3.1. *For some $0 < \epsilon, \delta < 1/2$, all $n \in \mathbb{N}$, $m := \lceil \epsilon n \rceil$ and $|t| \leq \epsilon\sqrt{n}$,*

$$\mathbb{P}(m \leq \mathcal{K}_t \leq n - m) \geq \delta\,.$$

Proof. First, observe that for $|t| \leq \epsilon\sqrt{n}$ by the definition of $\mathcal{K}_t$ and $S_j^{(2)}(t)$,

$$\begin{aligned}\mathbb{P}(\mathcal{K}_t < m) &\leq \mathbb{P}(\max_{0\leq j<m} S_j^{(2)}(t) \geq 2m\sqrt{n}) + \mathbb{P}(\max_{0\leq j\leq n} S_j^{(2)}(t) \leq 2m\sqrt{n}) \\ &\leq \mathbb{P}(\max_{0\leq j\leq m} S_j^{(2)} \geq \epsilon^{-1/2} m^{3/2}) + \mathbb{P}(\max_{0\leq j\leq n} S_j^{(2)} \leq 4\epsilon n^{3/2})\,.\end{aligned}$$

For $b = \mathbb{E}X_1^2$ finite and positive, by Donsker's invariance principle, $n^{-3/2}\max_{0\leq j\leq n} S_j^{(2)}$ converge in law as $n \to \infty$ to $\sqrt{b}\sup_{u\in[0,1]} Y(u)$. Hence, by (12), we deduce that

$$\lim_{\epsilon\downarrow 0}\lim_{n\to\infty} \mathbb{P}(\mathcal{K}_t < m) = 0 \text{ uniformly for all } |t| \leq \epsilon\sqrt{n}\,. \tag{28}$$

It remains to bound below $\mathbb{P}(\mathcal{K}_t \leq n - m)$. To this end, note that for $1 \leq j \leq m$,

$$S_{j+n-m}^{(2)}(t) = S_{n-m}^{(2)}(t) + jt + jS_{n-m} + \widetilde{S}_j^{(2)}\,,$$

where $\widetilde{S}_j^{(2)} = \sum_{i=1}^{j} \widetilde{S}_i$ and $\widetilde{S}_i = \sum_{\ell=1}^{i} X_{\ell+n-m}$. Hence, for $|t| \leq \epsilon\sqrt{n}$,

$$\begin{aligned}\mathbb{P}(\mathcal{K}_t \leq n - m) &\geq \mathbb{P}(\mathcal{K}_t \leq n - m, S_{n-m} \leq -2\sqrt{n}) \\ &\geq \mathbb{P}(S_{n-m} \leq -2\sqrt{n})\,\mathbb{P}(\max_{1\leq j\leq m}\{\widetilde{S}_j^{(2)} - j\sqrt{n}\} < 0)\,.\end{aligned}$$

Clearly, if $\widetilde{S}_i < \sqrt{n}$ for all i then necessarily $\widetilde{S}_j^{(2)} < j\sqrt{n}$, from which we deduce that for any $m \leq n/2$,

$$\mathbb{P}(\mathcal{K}_t \leq n - m) \geq \inf_{k\in[n/2,n]} \mathbb{P}(S_k \leq -2\sqrt{n})\,\mathbb{P}(\max_{1\leq j\leq n}\{S_j\} < \sqrt{n})\,. \tag{29}$$

Since $n^{-1/2}\max\{S_j : 1 \leq j \leq n\}$ converges in law to $\sqrt{b}$ times the absolute value of a standard Gaussian variable, we conclude that as $n \to \infty$, the right side of (29) remains bounded away from zero, which in view of (28) yields our thesis. ∎

By Lemma 3.1 we have that for $m = \lceil \epsilon n \rceil$ and $n \in \mathbb{N}$,

$$\delta\sqrt{\epsilon}\sqrt{m} \le 2\delta\epsilon\sqrt{n} \le \sum_{k=m}^{n-m} \mathbb{E}\Big[\int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} 1_{\{\mathcal{K}_t = k\}} dt\Big] \le \sum_{k=m}^{n-m} b_k \,.$$

Further, the contribution to (24) from $x \in [L, \infty)$ is at most

$$\int_L^\infty \mathbb{P}(-X_1 > x) dx = \mathbb{E}[(X_1 + L)^-] \le L^{-1}\mathbb{E}[X_1^2 1_{\{X_1^- \ge L\}}] \,.$$

With M as in the preceding bound on c_2, set $L = L(m) = \sqrt{m}/(2M)$, noting that the total contribution of these integrals to $\sum_{k=m}^{n-m} b_k$ is then at most

$$\frac{2M}{\epsilon}\sqrt{m}\mathbb{E}[X_1^2 1_{\{X_1^- \ge L(m)\}}] \,,$$

which for some $m_0 = m_0(\epsilon, \delta, M)$ finite and all $m \ge m_0$ is further bounded above by $(\delta/2)\sqrt{\epsilon}\sqrt{m}$. Consequently, setting

$$\kappa_m := \int_0^{L(m)} \mathbb{P}(-X_1 > x) g(x)^{-2} dx \,,$$

we get by monotonicity of $k \mapsto p_k^{(2)}$ and the arguments leading to (25), that for $m \ge m_0$,

$$\frac{\delta}{2}\sqrt{\epsilon}\sqrt{m} \le \kappa_m \sum_{k=m}^{n-m} p_k^{(2)} p_{n-k}^{(2)} \le \kappa_m \frac{m}{\epsilon}(p_m^{(2)})^2 \,.$$

Setting now $p_\ell^{(2)} := \ell^{-1/4}\psi(\ell)^{-1/2}$, we deduce from the preceding that

$$\kappa_m \ge \frac{\delta}{2}\epsilon^{3/2}\psi(m) \qquad \forall m \ge m_0 \,. \tag{30}$$

Now, by the same argument used for bounding c_2, we have that

$$g(x) \ge \eta p^{(2)}_{\lceil Mx \rceil^2} \ge \eta(1 + Mx)^{-1/2}\psi(\lceil Mx \rceil^2)^{-1/2} \,.$$

Fixing y and increasing m_0 as needed, for $m \ge m_0$ both $y \le L(m)$ and $\lceil ML(m) \rceil^2 \le m$. Hence, with $I(y, z) := \int_y^z (1 + Mx)\mathbb{P}(-X_1 > x)dx$ and $\psi_\star(r) := \sup_{\ell \le r} \psi(\ell)$, it follows that for $m \ge m_0$,

$$\begin{aligned}\eta^2\kappa_m &\le \int_0^{L(m)} (1 + Mx)\mathbb{P}(-X_1 > x)\psi(\lceil Mx \rceil^2)dx \\ &\le C(y) + I(y, \infty)\psi_\star(m) \,,\end{aligned}$$

where $C(y) := \psi_\star((1 + My)^2)I(0, y)$ is finite for any y finite. Considering this inequality and (30), we conclude that for some $c = c(\delta, \epsilon, M, \eta)$ positive, any y finite and all $m \ge m_0$ for which $\psi(m) = \psi_\star(m)$,

$$c\psi_\star(m) \le \eta^2\kappa_m \le C(y) + I(y, \infty)\psi_\star(m) \,.$$

Finally, with $\mathbb{E}[(1 + MX_1^-)^2]$ finite, clearly $I(y, \infty) \to 0$ as $y \to \infty$, hence the preceding inequality implies that $m \mapsto \psi(m)$ is bounded above. That is, $p_m^{(2)} \ge c_3 m^{-1/4}$ for some $c_3 > 0$ and all $m \ge 1$, as claimed in (8).

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